

FERMILAB-Pub-95/074-A

astro-ph/9504051

April 1995

Appearing in *Journal of Fluid Mechanics* **312**,327-340.

# Ekman pumping in compact astrophysical bodies

By MARK ABNEY<sup>1,2</sup> & RICHARD I. EPSTEIN<sup>2</sup>

<sup>1</sup>Department of Astronomy and Astrophysics, The Enrico Fermi Institute,  
The University of Chicago, Chicago, Illinois 60637  
and

NASA/Fermilab Astrophysics Center,  
Fermi National Accelerator Laboratory, Batavia, Illinois 60510

<sup>2</sup>Los Alamos National Laboratory,  
MS D436, Los Alamos, New Mexico 87545

We examine the dynamics of a rotating viscous fluid following an abrupt change in the angular velocity of the solid bounding surface. We include the effects of a density stratification and compressibility which are important in astrophysical objects such as neutron stars. We confirm and extend the conclusions of previous studies that stratification restricts the Ekman pumping process to a relatively thin layer near the boundary, leaving much of the interior fluid unaffected. We find that finite compressibility further inhibits Ekman pumping by decreasing the extent of the pumped layer and by increasing the time for spin-up. The results of this paper are important for interpreting the spin period discontinuities (“glitches”) observed in rotating neutron stars.

email: <sup>1,2</sup>abney@oddjob.uchicago.edu

<sup>2</sup>epstein@lanl.gov

# 1 Introduction

The approach to solid-body rotation of a fluid inside a rotating boundary is a familiar phenomenon with many applications. For instance, not only can we directly observe this phenomenon in the laboratory, but it may also play an important role in solar models, neutron stars and other environments. Greenspan & Howard (1963) give a fundamental analysis of the linearized version of this problem by considering a rotating axisymmetric container filled with a viscous incompressible fluid. They examine the behaviour of the fluid after the angular velocity of the container is suddenly changed by a small amount. Their solution consists of three distinct, time-separated phases: boundary layer formation, Ekman pumping and viscous relaxation. The bulk of the fluid spin-up (or down) occurs through Ekman pumping. Subsequent studies of the effect a stratification in density has on the Ekman pumping process (Walin 1969; Sakurai 1969; Buzyna & Veronis 1971; Hyun 1983; Spence, Foster & Davies 1992) found that some regions of the interior do not reach the final angular velocity until the viscous diffusion time has elapsed. The basic three stage structure of spin-up, however, remains intact where the intermediate Ekman pumping time results in a quasi-steady state in the fluid interior (see the review of Benton & Clark 1974 for a complete discussion of this subject).

The spin-up of a compressible fluid has been studied in the context of rapidly rotating gases (Sakurai & Matsuda 1974; Bark, Meijer & Cohen 1978; Lindblad, Bark, & Zahrai 1994). These studies concluded that the basic structure for spin-up remained unchanged, but with some important and interesting differences. Numerical studies have also been carried out for both the linear and non-linear regimes (Hyun & Park 1990, 1992; Park & Hyun 1994) with the results lending support to the analytical results. The compressible fluid research, however, was motivated by a

desire to understand the dynamics of gas centrifuges where the effects of gravity are negligible compared to centrifugal forces. A density gradient results, but in a direction perpendicular to the angular velocity.

In this investigation, we are primarily concerned with the possible astrophysical applications of the theory of stratified, compressible rotating fluids. Indeed one of the primary motivations for studying spin-up of a stratified fluid was to understand the sun's rotation (Howard, Moore & Spiegel 1967; Sakurai, Clark & Clark 1971; Clark, Clark, Thomas & Lee 1971). Though the sun does not have a solid outer crust, Ekman suction may arise because of solar wind torque, or boundary layer flow taking place between fluid interfaces. Ekman pumping in multilayer fluids was investigated by Pedlosky (1967) with his theory tested experimentally by Linden & van Heijst (1984) and O'Donnell & Linden (1992). Recently, the possibility that Ekman pumping may play a role in the synchronization of some binary stars has also been discussed (Tassoul & Tassoul 1990, 1992; Rieutord 1992). We, however, are motivated by the phenomenon of pulsar "glitches," a sudden slight increase in a neutron star's rotational frequency, and the resultant response of the fluid core. The simplified model of a rotating neutron star we consider includes the effect of compressibility of the inner fluid as well as a density gradient parallel to the gravitational field and angular velocity. A strong magnetic field, as exists on the surface, may alter the internal dynamics, but the presence of a magnetic field in the core is not well established and we choose to ignore it. The effects of a magnetic field which threads the core are discussed in greater detail in a forthcoming paper.

## 2 Ekman pumping

### *2.1 Fluid dynamics*

To investigate the response of the fluid in a rotating container, we examine the usual, simple model of a cylinder of height  $2L_*$  and radius  $r_{c*}$  rotating with angular velocity  $\boldsymbol{\Omega}_*$  (here and elsewhere an asterisk subscript indicates a dimensional variable or operator; quantities without this subscript are dimensionless.) When the angular velocity of the container is abruptly changed by a small amount, the differential rotation between the fluid and the top and bottom of the cylinder generates the “Ekman pumping” process. Unlike previous studies, we introduce an equation of state relating the mass-energy density  $\rho_*$  to the pressure  $p_*$  and to the composition. For a fluid with a kinematic viscosity  $\nu_*$ , the Navier–Stokes equations of motion in a frame rotating with angular velocity  $\boldsymbol{\Omega}_*$  are

$$\rho_* \left( \frac{\partial \mathbf{v}_*}{\partial t_*} + \mathbf{v}_* \cdot \nabla_* \mathbf{v}_* + 2(\boldsymbol{\Omega}_* \times \mathbf{v}_*) \right) = -\nabla_* p_* + \rho_* \mathbf{g}_* + \frac{1}{2} \rho_* \nabla_* \Omega_*^2 r_*^2 + \rho_* \nu_* \nabla_*^2 \mathbf{v}_*, \quad (1)$$

where  $\mathbf{g}_*$  is,

$$\mathbf{g}_* = \begin{cases} -g_* \mathbf{e}_z & z_* > 0 \\ +g_* \mathbf{e}_z & z_* < 0 \end{cases}, \quad (2)$$

$r_*$  is the cylindrical radius, and we take  $\boldsymbol{\Omega}_* = \Omega_* \mathbf{e}_z$  with  $\mathbf{e}_z$  the unit vector in the  $z$ -direction. From this point onward we consider only the upper half-plane, noting that all quantities are symmetric about  $z_* = 0$ .

This particular form of  $\mathbf{g}_*$  represents a constant inward pointing acceleration, even though in a self-gravitating body the acceleration is a decreasing function of the radius, becoming zero at the center. However, as we will show below, for the parameter ranges applicable to a neutron star the significant dynamics occurs in a thin layer near the boundaries where gradients to the gravitational acceleration are negligible.

As long as  $r_{c*}$  is not too large the centrifugal acceleration is small compared to the gravitational acceleration and can be neglected. More precisely, we assume that finite Froude number effects can be ignored, i.e.  $F \equiv 4\Omega_*^2 r_{c*} / g_* \ll 1$ , where  $F$  is the Froude

number. This results in a state of rotational equilibrium where the pressure  $p_{s*}$  and density  $\rho_{s*}$  are functions only of  $z_*$ . The Navier–Stokes equation for the equilibrium system is

$$\frac{\partial}{\partial z_*} p_{s*} = -\rho_{s*} g_* \quad (3)$$

We now look at a perturbed system in which the angular velocity of the boundary is suddenly changed by a relatively small amount  $\Delta\Omega_*$ . The resulting pressure and density are

$$p_* = p_{s*}(z_*) + \delta p_*(r_*, z_*, t_*) \quad (4)$$

$$\rho_* = \rho_{s*}(z_*) + \delta \rho_*(r_*, z_*, t_*). \quad (5)$$

To first order in  $\delta p_*$ ,  $\delta \rho_*$  and  $v_*(r_*, z_*, t_*)$  we have

$$\frac{\partial \mathbf{v}_*}{\partial t_*} + 2\Omega_* \mathbf{e}_z \times \mathbf{v}_* = -\frac{1}{\rho_{s*}} \nabla_* \delta p_* - \frac{1}{\rho_{s*}} \delta \rho_* \mathbf{g}_* + \nu_* \nabla_*^2 \mathbf{v}_*. \quad (6)$$

We non-dimensionalize the equations by writing variables and operators as a dimensional constant times a non-dimensional variable or operator as follows,

$$\mathbf{v}_* \equiv (L_* \Delta\Omega_*) \mathbf{v}$$

$$t_* \equiv (E^{1/2} 2\Omega_*)^{-1} t$$

$$\mathbf{r}_* \equiv L_* \mathbf{r}$$

$$\mathbf{z}_* \equiv L_* z \mathbf{e}_z$$

$$\delta p_* \equiv (2\Omega_* \rho_{0*} L_*^2 \Delta\Omega_*) \delta p$$

$$\delta \rho_* \equiv (2\Omega_* \rho_{0*} L_* \Delta\Omega_* / g_*) \delta \rho$$

$$\rho_{s*} \equiv \rho_{0*} \rho_s$$

$$\nabla_* \equiv (1/L_*) \nabla$$

where  $\rho_{0*}$  is a fiducial value for the equilibrium density. We also introduce the dimensionless viscosity, or Ekman number,

$$E = \frac{\nu_*}{2\Omega_* L_*^2}. \quad (7)$$

The Navier–Stokes equation for the perturbations is now

$$E^{1/2} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{e}_z \times \mathbf{v} = -\frac{1}{\rho_s} \nabla \delta p - \frac{1}{\rho_s} \delta \rho \mathbf{e}_z + E \nabla^2 \mathbf{v}, \quad (8)$$

or in terms of the individual cylindrical components,

$$E^{1/2} \frac{\partial u}{\partial t} - v = -\frac{\partial}{\partial r} \frac{\delta p}{\rho_s} + E \left( \nabla^2 - \frac{1}{r^2} \right) u \quad (9)$$

$$E^{1/2} \frac{\partial v}{\partial t} + u = E \left( \nabla^2 - \frac{1}{r^2} \right) v \quad (10)$$

$$E^{1/2} \frac{\partial w}{\partial t} = -\frac{1}{\rho_s} \frac{\partial}{\partial z} \delta p - \frac{\delta \rho}{\rho_s} + E \nabla^2 w, \quad (11)$$

where  $(u, v, w)$  are the velocities in the  $(r, \theta, z)$  directions. We need two more equations in order to complete the formulation of the problem, an equation of state and the continuity equation.

We describe the fluid in terms of the pressure and the concentrations of its constituent elements. Within the context of neutron stars these elements are mainly electrons, protons and neutrons. The equation of state then relates the density to these quantities,  $\rho_* = \rho_*(p_*, Y_i)$  where  $Y_i$  is the concentration of the  $i$ -th particle species<sup>1</sup>.

The nature of the restoring force and the corresponding Brunt–Väisälä frequency is most readily calculated in the Lagrangian, as opposed to the Eulerian, formulation

---

<sup>1</sup>In the core of an equilibrium neutron star the  $Y_i$  are the concentrations that minimize the free energy through nuclear and weak interaction reactions. In the perturbations considered here, the fluctuation time scales are short compared to those for the weak interactions to adjust the ratio of neutrons to proton. The values of the  $Y_i$  can thus be considered as fixed properties of the matter. If the equilibrium values of  $Y_i$  give a stable stratification, buoyant forces will cause perturbations to oscillate with the Brunt–Väisälä frequency.

of the perturbations. We use  $\delta q_*$  for an Eulerian perturbation of a quantity  $q_*$ , the difference between the actual and non-perturbed values of that quantity at a given point in space and time. A Lagrangian perturbation  $\Delta q_*$  describes the change from the non-perturbed value an element of fluid experiences as it travels from one point to another. The two perturbations are related by a displacement vector field,  $\boldsymbol{\xi}_*$ ,

$$\Delta q_* = \delta q_* + \boldsymbol{\xi}_* \cdot \nabla_* q_{0*}, \quad (12)$$

where  $q_{0*}(\mathbf{r})$  is the non-perturbed quantity. The displacement vector field  $\boldsymbol{\xi}_*$  is related to the velocity by,

$$\mathbf{v}_* = \frac{\partial}{\partial t_*} \boldsymbol{\xi}_*. \quad (13)$$

In non-dimensional notation  $\boldsymbol{\xi}_* = (L_* \Delta \Omega_* / 2 \Omega_*) \boldsymbol{\xi}$  and

$$\mathbf{v} = E^{1/2} \frac{\partial}{\partial t} \boldsymbol{\xi}. \quad (14)$$

To relate the density and pressure perturbations, consider a fluid displacement in which some quantity  $Y$  is held constant i.e.  $\Delta Y = 0$ . The Lagrangian perturbations  $\Delta \rho_*$  and  $\Delta p_*$  are then related by

$$\Delta \rho_* = \left( \frac{\partial \rho_*}{\partial p_*} \right)_Y \Delta p_* \equiv \frac{1}{c_{Y*}^2} \Delta p_*. \quad (15)$$

If the fluid is displaced adiabatically so that the entropy and composition are fixed, then  $c_{Y*}$  is the usual sound speed. We characterize the equilibrium relationship between the density and the pressure by

$$\frac{\partial \rho_{s*} / \partial z_*}{\partial p_{s*} / \partial z_*} = \left( \frac{\partial \rho_*}{\partial p_*} \right)_{\text{eq}} \equiv \frac{1}{c_{\text{eq}*}^2}. \quad (16)$$

With (12)–(16) we can relate  $\delta \rho_*$  and  $\delta p_*$ :

$$\delta \rho_* = \Delta \rho_* - \boldsymbol{\xi}_* \cdot \nabla \rho_{s*} \quad (17)$$

$$= \frac{1}{c_{Y*}^2} \delta p_* + \left( \frac{1}{c_{\text{eq}*}^2} - \frac{1}{c_{Y*}^2} \right) \rho_{s*} g_* \xi_{z*}, \quad (18)$$

with  $\xi_{z*}$  the  $z$ -component of  $\boldsymbol{\xi}_*$ . Once again, non-dimensionalizing we obtain,

$$\delta\rho = \left(\frac{g_* L_*}{c_{Y*}^2}\right) \delta p + \left(\frac{N_*^2}{4\Omega_*^2}\right) \rho_s \xi_z \equiv \kappa_Y \delta p + N^2 \rho_s \xi_z, \quad (19)$$

where the Brunt–Väisälä frequency  $N_*$  is

$$N_*^2 \equiv g_*^2 \left( \frac{1}{c_{\text{eq}*}^2} - \frac{1}{c_{Y*}^2} \right), \quad (20)$$

and the two dimensionless parameters  $\kappa_Y \equiv g_* L_* / c_{Y*}^2$  and  $N = N_* / 2\Omega_*$  are the “constant- $Y$  compressibility” and the normalized Brunt–Väisälä frequency, respectively. In previous studies  $\kappa_Y$  was assumed to be negligible, but in self gravitating astronomical bodies  $\kappa_Y$  can be of order unity or much larger. Returning to our example of the neutron star, for instance, we can estimate the size of  $\kappa_Y$ . Using the values  $g_* \approx 10^{14} \text{cm/sec}^2$ ,  $L_* \approx 10^6 \text{cm}$  and  $c_{Y*} \approx 10^9 \text{cm/sec}$  (Epstein 1988), we obtain  $\kappa_Y \approx 10^2$ .  $N$  characterizes the influence of density stratification on Ekman pumping.

At this point, it is worth noting that (18) is a generalization of more familiar formulas for low frequency density perturbations in a compressible stratified fluid. In particular, in studies of the terrestrial atmosphere one usually assumes that the fluid motions are slow enough to allow the pressure of displaced fluid to adjust to its surroundings so that  $\delta p = 0$  and the first term in the left hand side of (18) vanishes. This is equivalent to what we find in §6.4 of Pedlosky (1979). Furthermore, if we consider the incompressible limit,  $c_{Y*}^2 \rightarrow \infty$ , then

$$N_*^2 \rightarrow -\frac{g_*}{\rho_*} \frac{\partial \rho_*}{\partial z_*}, \quad (21)$$

which is the familiar incompressible formula for the Brunt–Väisälä frequency.

The final equation is the continuity equation for the perturbations,

$$\frac{\Delta \rho_*}{\rho_{s*}} = -\nabla_* \cdot \boldsymbol{\xi}_* \quad (22)$$

$$= -\nabla_{r*} \cdot \boldsymbol{\xi}_{r*} - \frac{\partial \xi_{z*}}{\partial z_*}. \quad (23)$$

With (12), the continuity equation becomes,

$$\delta\rho_* + \xi_{z*} \frac{\partial \rho_{s*}}{\partial z_*} + \rho_{s*} \nabla_{r*} \cdot \xi_{r*} + \rho_{s*} \frac{\partial \xi_{z*}}{\partial z_*} = 0. \quad (24)$$

Using (16) and (18), and taking the time derivative of (24), we get

$$\frac{1}{\rho_{s*} c_{Y*}^2} \frac{\partial}{\partial t_*} \delta p_* - \frac{g_*}{c_{Y*}^2} w_* + \nabla_{r*} \cdot u_* + \frac{\partial w_*}{\partial z_*} = 0. \quad (25)$$

In non-dimensionalized units this is

$$E^{1/2} \Omega^2 \frac{\partial}{\partial t} \frac{\delta p}{\rho_s} - \kappa_Y w + \frac{\partial w}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (ru) = 0, \quad (26)$$

where we have introduced the dimensionless angular velocity,

$$\Omega \equiv \frac{2L_* \Omega_*}{c_{Y*}}. \quad (27)$$

We can now rearrange the complete set of perturbation equations in a more convenient form. We use (19) to eliminate  $\delta\rho$  in (11) and take the time derivative to obtain,

$$E \frac{\partial^2 w}{\partial t^2} = -N^2 w - E^{1/2} \frac{\partial}{\partial z} \frac{\partial}{\partial t} \frac{\delta p}{\rho_s} - E^{1/2} (\kappa_Y - \kappa_{\text{eq}}) \frac{\partial}{\partial t} \frac{\delta p}{\rho_s} + E^{3/2} \nabla^2 \frac{\partial}{\partial t} w. \quad (28)$$

where the “equilibrium compressibility” is

$$\kappa_{\text{eq}} \equiv \frac{g_* L_*}{c_{\text{eq}*}^2}. \quad (29)$$

Since the constant- $Y$  and equilibrium compressibilities are comparable, we write

$$\Delta\kappa \equiv \kappa_{\text{eq}} - \kappa_Y = \frac{N_*^2 L_*}{g_*} = F N^2 \frac{L_*}{r_{c*}} \ll 1. \quad (30)$$

The variable  $\delta p$  only occurs in the combination  $\delta p/\rho_s$ , so we define  $\delta P \equiv \delta p/\rho_s$ . The final equations are now

$$E^{1/2} \frac{\partial}{\partial t} u - v = -\frac{\partial}{\partial r} \delta P + E \left( \nabla^2 - \frac{1}{r^2} \right) u \quad (31)$$

$$E^{1/2} \frac{\partial}{\partial t} v + u = E \left( \nabla^2 - \frac{1}{r^2} \right) v \quad (32)$$

$$E \frac{\partial^2 w}{\partial t^2} = -N^2 w - E^{1/2} \frac{\partial}{\partial z} \frac{\partial}{\partial t} \delta P + E^{1/2} \Delta \kappa \frac{\partial}{\partial t} \delta P + E^{3/2} \nabla^2 \frac{\partial}{\partial t} w. \quad (33)$$

$$E^{1/2} \Omega^2 \frac{\partial}{\partial t} \delta P - \kappa_Y w + \frac{\partial}{\partial z} w + \frac{1}{r} \frac{\partial}{\partial r} (ru) = 0. \quad (34)$$

Note the harmonic restoring force provided by the Brünt–Väisälä term in (33). The above four equations, (31)–(34), describe the evolution of the four unknowns,  $\mathbf{v}$  and  $\delta P$ . We have dimensionless parameters,  $E$ ,  $N^2$ ,  $\Delta \kappa$ ,  $\kappa_Y$  and  $\Omega^2$ . In order to reduce the parameter space we consider only slow rotation,  $\Omega^2 \ll 1$ , and  $\Delta \kappa \ll 1$ . Both  $\Omega^2$  and  $\Delta \kappa$  are easily included in the general solution, but since they only appear as a product with  $E^{1/2}$  their effects are small and we do not consider these terms in what follows. Table 1 contains definitions of the dimensionless parameters and representative values for a neutron star.

The presence of the compressibility  $\kappa_Y$  distinguishes this set of equations from earlier studies (Walin 1969; Sakurai 1969; Clark *et al.* 1971). Previous studies chose to emphasize the effects of temperature on the density of the fluid. Specifically, the density was considered a function of the temperature and the stratification was a result of a temperature gradient which was imposed by the boundary conditions. The dynamical significance of the stratification and Brunt–Väisälä frequency arose through the effects of temperature diffusion and the heat equation. This approach is not appropriate to the astrophysical cases with which we are primarily concerned. In neutron stars, for example, thermal effects have a negligible result on the fluid dynamics, whereas compressibility is quite significant. We, therefore, focus on the dependence on the equation of state.

## 2.2 Boundary values and initial conditions

To obtain a unique solution to (31)–(34), we need to specify both the boundary and initial conditions to our problem. There are, in essence, two approaches to take at this point. The most complete method is to state that initially the fluid rotates uniformly with the cylinder, and solve for the behaviour of the fluid after the angular velocity of the cylinder changes with the Laplace transformation technique (Greenspan & Howard 1963). A simpler, and more physically elucidating approach, although less rigorous, used by other researchers in the field (Walsh 1969; Sakurai 1969; Barcilon & Pedlosky 1967) entails recognizing that different physical processes take place on widely different time scales in different regions of the fluid. We will follow this latter approach.

If the Ekman number  $E$ , or dimensionless viscosity, is sufficiently small, the behaviour of the fluid following an abrupt change in rotation rate of the container can be viewed as three distinct physical processes which occur on time scales  $\Omega_*^{-1}$ ,  $E^{-1/2}\Omega_*^{-1}$  and  $E^{-1}\Omega_*^{-1}$ . The most rapid process is the formation of a viscous boundary layer. Following the impulsive change of rotation of the cylinder, a viscous Rayleigh shear layer forms on the upper and lower surfaces in a time scale on the order of a rotation time ( $t_{b*} \approx \Omega_*^{-1}$ ). Within this region the gradient in the azimuthal velocity results in an imbalance between the centrifugal and pressure gradient forces causing fluid to flow radially. This radial flow in the boundary layer establishes a secondary flow where fluid in the interior is pulled into the boundary layer to replace the flow in the Ekman layer, creating an opposing radial flow in the interior fluid that satisfies continuity requirements. This Ekman pumping spins the interior of the fluid up in a time scale of order  $E^{-1/2}\Omega_*^{-1}$ . With our choice of dimensionless variables this corresponds to a dimensionless time,  $t_E \approx 1$ . Finally, residual oscillations decay in the viscous diffusion time  $t_{v*} \approx E^{-1}\Omega_*^{-1}$ .

Since the principal goal of this investigation is to understand the effects of the

stratification and compressibility on the Ekman pumping in the interior of the fluid, we expand (31)–(34) in powers of  $E^{1/2}$  and isolate the equations relating to Ekman pumping. The initial velocity distribution for the Ekman pumping equation is equivalent to the final velocity distribution of the boundary layer which forms during the first phase. Following Walin (1969), we formulate the boundary condition in terms of the continuity of the velocity perpendicular to the Ekman boundary just outside the boundary layer,

$$w(z = \pm 1) = \mp \frac{E^{1/2}}{\sqrt{2}} (\nabla \times \mathbf{v})_z \quad (35)$$

$$= \mp \frac{E^{1/2}}{\sqrt{2}} \frac{1}{r} \frac{\partial}{\partial r} (rv). \quad (36)$$

It is critical to the dynamics of Ekman pumping that the vertical velocity at the boundary layer is  $O(E^{1/2})$ . This standard result (see, e.g. Pedlosky 1979) can be understood by scaling arguments. The imbalance between the centrifugal forces and pressure gradient forces in the boundary layer drives the Ekman pumping process. The thickness  $\lambda$  of the boundary layer is  $O(E^{1/2})$  since the viscous terms in the dimensionless Navier–Stokes equation is  $E\nabla^2 \approx E/\lambda^2 = O(1)$ . The mass flux *within* the boundary layer is  $\dot{M}_\lambda \propto \lambda = O(E^{1/2})$ . The net mass flux  $\dot{M}_z \propto w$  *perpendicular* to the boundary layer is of the same order as  $\dot{M}_\lambda$  giving  $w = O(E^{1/2})$ . The sidewalls also have an  $O(E^{1/2})$  boundary layer. As shown in Pedlosky (1967) the vertical velocity through this layer is inhibited by buoyancy forces and is  $O(E^{1/2})$  leading to a net mass flux of  $O(E)$ . The boundary condition of the radial velocity is then  $u = O(E)$  at  $r_c$ . Since we keep terms only up to  $O(E^{1/2})$ , as described in the following section, this gives  $u(r_c) = 0$ .

### 3 Solutions

To solve (31)–(34) perturbatively we expand each fluid variable  $q$  as a series  $q = q_0 + E^{1/2}q_1 + Eq_2 + \dots$ . Collecting terms of a given power of  $E^{1/2}$ , we obtain a set of equations governing each order in the expansion.

We find that the  $O(1)$  equations are,

$$v_0 = \frac{\partial}{\partial r} \delta P_0 \quad (37)$$

$$u_0 = 0 \quad (38)$$

$$w_0 = 0 \quad (39)$$

$$-\kappa_Y w_0 + \frac{\partial}{\partial z} w_0 + \frac{1}{r} \frac{\partial}{\partial r} (r u_0) = 0, \quad (40)$$

and the  $O(E^{1/2})$  equations are,

$$v_1 = \frac{\partial}{\partial r} \delta P_1 \quad (41)$$

$$\frac{\partial}{\partial t} v_0 = -u_1 \quad (42)$$

$$N^2 w_1 = -\frac{\partial}{\partial z} \frac{\partial}{\partial t} \delta P_0 \quad (43)$$

$$-\kappa_Y w_1 + \frac{\partial}{\partial z} w_1 + \frac{1}{r} \frac{\partial}{\partial r} (r u_1) = 0. \quad (44)$$

We define  $\phi \equiv -\partial \delta P_0 / \partial t$ , so that (37) and (42) become

$$u_1 = \frac{\partial}{\partial r} \phi \quad (45)$$

and (43) and (44) are now, respectively,

$$N^2 w_1 = \frac{\partial}{\partial z} \phi \quad (46)$$

$$\frac{-\kappa_Y}{N^2} \left( \frac{\partial}{\partial z} \phi \right) + \frac{\partial}{\partial z} \frac{1}{N^2} \left( \frac{\partial}{\partial z} \phi \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \phi \right) = 0. \quad (47)$$

Assuming  $N^2$  varies slowly over  $z$ , we treat it as a constant and simplify (47) to

$$\frac{\partial^2}{\partial z^2}\phi - \kappa_Y \frac{\partial}{\partial z}\phi + N^2 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \phi \right) = 0. \quad (48)$$

After taking the time derivative, the boundary condition, (36), is

$$\frac{\partial}{\partial t} \frac{\partial}{\partial z} \phi = \pm \frac{N^2}{\sqrt{2}} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \phi \right), \quad \text{at } z = \pm 1. \quad (49)$$

By setting  $\phi = Z(z) R(r) T(t)$ , (48) becomes,

$$\frac{1}{Z} \frac{d^2}{dz^2} Z - \frac{\kappa_Y}{Z} \frac{d}{dz} Z + \frac{N^2}{R} \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} R \right) = 0. \quad (50)$$

The solutions to the spatial functions are

$$Z = A e^{\beta_+ z} + B e^{\beta_- z} \quad (51)$$

$$R = J_0(kr), \quad (52)$$

where

$$\beta_{\pm} = \frac{1}{2} \left( \kappa_Y \pm (\kappa_Y^2 + 4k^2 N^2)^{\frac{1}{2}} \right) \quad (53)$$

The symmetry of the boundary condition,  $w_1(z=0) = 0$ , relates  $A$  and  $B$ :

$$A = -\frac{\beta_-}{\beta_+} B. \quad (54)$$

The constant  $B$  is arbitrary, and we choose it so that  $Z(1) = 1$ . This leads to

$$B = \left( e^{\beta_-} - \frac{\beta_-}{\beta_+} e^{\beta_+} \right)^{-1}. \quad (55)$$

The possible values of  $k$  are determined by the boundary condition at the sidewall at  $r = r_c$ , i.e.  $u_1(r_c) = 0$ . From (45) and (52) we see that this condition corresponds to  $J_1(k_m r_c) = 0$  for  $m = 0, 1, 2, \dots$ . The first zeros of  $J_1$  are  $k_m r_c = 0, 3.8317, 7.0156, \dots$ . The solution  $k_m = 0$  has  $\mathbf{v} = 0$  everywhere and is of no interest.

We utilize the boundary condition to determine the time dependence of  $\phi$ . Putting our solution for  $R$  and  $Z$  into (36), we obtain the differential equation,

$$\frac{d}{dt}T = \frac{-k^2 N^2}{\sqrt{2}} \frac{Z(1)}{\frac{dZ}{dz}(1)} T, \quad (56)$$

whose solution is,

$$T(t) = e^{-\omega t}, \quad (57)$$

where,

$$\omega = \frac{k^2 N^2}{\sqrt{2}} \left( A\beta_+ e^{\beta_+} + B\beta_- e^{\beta_-} \right)^{-1}, \quad (58)$$

with  $A$  and  $B$  defined as above.

There are two interesting limiting cases. The first is that of no stratification  $N \rightarrow 0$ ; the second is that of an incompressible fluid  $\kappa_Y \rightarrow 0$ . Let us consider the first of these which gives,

$$\beta_+ \approx \kappa_Y \left( 1 + \frac{k_m^2 N^2}{\kappa_Y^2} \right) \quad (59)$$

$$\beta_- \approx -\frac{k_m^2 N^2}{\kappa_Y} \quad (60)$$

$$B \approx 1 + \frac{k_m^2 N^2}{\kappa_Y} (1 + e^{\kappa_Y} / \kappa_Y) \quad (61)$$

$$A \approx \frac{k_m^2 N^2}{\kappa_Y^2} \quad (62)$$

and,

$$\omega \approx \frac{\kappa_Y}{\sqrt{2}(e^{\kappa_Y} - 1)}. \quad (63)$$

Equation (63) shows that for large compressibilities the Ekman spin-up time scale  $\omega^{-1}$  grows exponentially with  $\kappa_Y$ . The second limiting case,  $\kappa_Y \rightarrow 0$ , gives  $\beta_{\pm} \approx \pm k_m N$  and

$$\omega \approx \frac{k_m N \cosh k_m N}{\sqrt{2} \sinh k_m N}. \quad (64)$$

This matches the  $\kappa_Y = 0$ ,  $N \neq 0$  solution which was obtained by Walin (1969).

We are now in a position to write the complete solution for the quantities  $\phi$  and  $v_0$ ;

$$\phi = \sum_{m=1}^{\infty} C_m Z_m(z) J_0(k_m r) e^{-\omega_m t}. \quad (65)$$

The velocity,  $v_0$ , is found from the relationship,

$$\frac{\partial}{\partial r} \phi = -\frac{\partial}{\partial t} v_0, \quad (66)$$

which gives,

$$v_0(r, z, t) = - \sum_{m=1}^{\infty} \frac{k_m}{\omega_m} C_m Z_m(z) J_1(k_m r) e^{-\omega_m t} + v_{\infty}(r, z). \quad (67)$$

The last term represents the final velocity due to Ekman pumping. If we take the frame of reference as that rotating with the cylinder before impulsive spin-up, the final velocity at the boundary of the interior fluid is

$$v_{\infty}(r, z = \pm 1) = r. \quad (68)$$

We determine  $C_m$  from the initial state of the fluid,

$$v_0(r, z = 1, t = 0) = 0 = - \sum_{m=1}^{\infty} \frac{k_m C_m}{\omega_m} J_1(k_m r) + r. \quad (69)$$

The coefficients, given by the standard equation for a Fourier–Bessel series, are

$$\frac{k_m C_m}{\omega_m} = \frac{2}{k_m J_2(k_m)}, \quad (70)$$

and the final velocity is

$$v_{\infty}(r, z) = 2 \sum_{m=1}^{\infty} \frac{1}{k_m J_2(k_m)} J_1(k_m r) Z_m(z), \quad (71)$$

where we have chosen  $r_c = 1$  ( $r_{c*} = L_*$ ).

## 4 Discussion

The time dependence of the Ekman pumping process is exponential with characteristic time  $1/\omega$ . We plot the value of  $\omega$  as a function of  $k_m N$  in figure 1 for different values of the parameter  $\kappa_Y$ . Larger  $N$ , corresponding to greater density stratification gives larger  $\omega$  and reduced characteristic time. That is, a strongly stratified fluid spins up much quicker than a non-stratified fluid. On the other hand, an increased value of the compressibility  $\kappa_Y$  slows the pumping process for a given  $k_m N$ . The spin-up time  $\omega^{-1}$  decreases with increased stratification because stratification isolates much of the fluid from the pumping process.

Figures 2 and 3 show that the rotation state at the end of the Ekman pumping stage is not that of a solid body. The ordinate  $Z(z)$  is proportional to the final azimuthal velocity, with  $Z = 1$  being the largest possible spin-up. Larger values of  $k_m N$  leave more of the internal fluid unaffected by the Ekman pumping process. In contrast, in a homogeneous fluid,  $N = 0$ , Ekman pumping brings the entire fluid to an angular velocity equal to that of the boundary. The compressibility  $\kappa_Y$  further decreases the amount of pumped fluid, as we can see by comparing figure 2, for  $\kappa_Y = 0$ , with figure 3, for  $\kappa_Y = 10$ .

Compressibility thus decreases the efficacy of Ekman pumping both by lengthening the spin-up time and by decreasing the amount of affected fluid. To convey a clearer picture of how strong the effect of  $\kappa_Y$  is, we plot in figure 4 the final angular velocity of the fluid at its central ( $z = 0$ ) layer as a function of  $k_m N$  for different values of  $\kappa_Y$ . Though there is little change between  $\kappa_Y = 0$  and  $\kappa_Y = 1$ , the internal final angular velocity is strongly suppressed as  $\kappa_Y$  increases to 10.

We point out that for canonical values  $N^2 \approx 6$  and  $\kappa_Y \approx 10^2$  for a neutron star, the thickness of the layer affected by Ekman pumping is much smaller than the radius

of the cylinder. Gradients in the gravitational acceleration are therefore small in these layers, justifying our original assumption of constant  $g_*$  in (2).

In figure 5 we plot the average spin-up of the fluid  $\langle Z \rangle$  as a function of the normalized Brunt–Väisälä frequency  $N$  for the two lowest order modes,  $k_1$  and  $k_2$ . We see that even modest values of  $N$  prevent most of the fluid from spinning up during the Ekman pumping phase. The state of the fluid after a time scale of  $t_* \approx E^{-1/2}\Omega_*^{-1}$  is, thus, one of non-uniform rotation. The process of viscous diffusion, which operates in a time  $t_{v*} \approx E^{-1}\Omega_*^{-1}$  eventually brings the fluid into solid-body rotation.

The case of spherical geometry was studied by Clark *et al.* (1971), where they found that the solution for a sphere is qualitatively similar to that of the cylinder. That is, the final state of non-uniform rotation also exists in the sphere, but the geometry of the layer that gets Ekman pumped is modified.

We find a particularly interesting application of these phenomena is the response of the interior of a rotating neutron star to a glitch, a sudden small change in the rotational velocity. Within the star there exists a significant stratification due to the strong gravitational field and the equilibrium concentrations of protons, neutrons and electrons. Reisenegger & Goldreich (1992) estimated a value of  $N_* \approx 500\text{s}^{-1}$  for a neutron star. For a canonical value of  $\Omega_* \approx 100\text{s}^{-1}$ , we obtain  $N \approx 2.5$ . This is large enough to have a significant effect on the length of time the core of the star needs to come into rotational equilibrium. We explore these issues in a forthcoming paper.

We would like to thank Angela Olinto for her valuable input and advice. M.A. was supported in part by the DOE at Chicago, by NASA grant NAG 5-2788 at Fermi National Laboratory, and through a collaborative research grant from IGPP/LANL. This work was carried out under the auspices of the U.S. Department of Energy.

## References

Barcilon, V. & Pedlosky, J. 1967, Linear theory of rotating stratified fluid motions.

- J. Fluid Mech.* **29**, 1.
- Bark, F. H., Meijer, P. S. & Cohen, H. I. 1978, Spin up of a rapidly rotating gas. *Physics of Fluids* **21**, 531–539.
- Benton, E. R. & Clark, A. 1974, Spin Up. *Ann. Rev. Fluid Mech.* **6**, 257–280.
- Buzyna, G. & Veronis, G. 1971, Spin-up of a stratified fluid: theory and experiment. *J. Fluid Mech.* **50**, 579–608.
- Clark, A., Clark, P. A., Thomas, J. H. & Lee N. 1971, Spin-up of a strongly stratified fluid in a sphere. *J. Fluid Mech.* **45**, 131–149.
- Epstein, R. I. 1988, Acoustic properties of neutron stars. *Astrophys. J.* **333**, 880–894.
- Greenspan, H. P. & Howard, L. N. 1963, On a time-dependent motion of a rotating fluid. *J. Fluid Mech.* **17**, 385–404.
- Howard, L. N., Moore, D. W. & Spiegel, E. A. 1967, Solar spin-down problem. *Nature* **214**, 1297–1299.
- Hyun, J. M. 1983, Axisymmetric flows in spin-up from rest of a stratified fluid in a cylinder. *Geophys. Astrophys. Fluid Dyn.* **23**, 127–141.
- Hyun, J. M. & Park, J. S. 1990, Early time behavior of the Ekman layers in spin-up of a rapidly rotating gas. *J. Phys. Soc. Japan* **59**, 3584–3594.
- Hyun, J. M. & Park, J. S. 1992, Spin-up from rest of a compressible fluid in a rapidly rotating cylinder. *J. Fluid Mech.* **237**, 413–434.
- Lindblad, I. A. A., Bark, F. H. & Zahrai, S. 1994, Spin-up of a rapidly rotating heavy gas in a thermally insulated annulus. *J. Fluid Mech.* **274**, 383–404.
- Linden, L. N. & van Heijst, G. J. F. 1984, Two-layer spin up and frontogenesis. *J. Fluid Mech.* **143**, 69–94.
- O'Donnell, J. & Linden, P. F. 1992, Spin-up of a two-layer fluid in a rotating cylinder. *Geophys. Astrophys. Fluid Dyn.* **66**, 47–66.
- Park, J. S. & Hyun, J. M. 1994, Dynamical structure of compressible fluid flows in an abruptly rotating cylinder. *J. Phys. Soc. Japan* **63**, 528–535.
- Pedlosky, J. 1967, The spin-up of a stratified fluid. *J. Fluid Mech.* **28**, 463–479.
- Pedlosky, J. 1979, *Geophysical Fluid Dynamics*, Springer.

- Reisenegger, A. & Goldreich, P. 1992, A new class of g-modes in neutron stars. *Astrophys. J.* **395**, 240–249.
- Rieutord, M. 1992, Ekman circulation and the synchronization of binary stars. *Astron. Astrophys.* **259**, 581–584.
- Sakurai, T. 1969, Spin down problem of a rotating stratified fluid in thermally insulated circular cylinders. *J. Fluid Mech.* **37**, 689–699.
- Sakurai, T., Clark, A. & Clark, P. A. 1971, Spin-down of a Boussinesq fluid of small Prandtl number in a circular cylinder. *J. Fluid Mech.* **49**, 753–773.
- Sakurai, T. & Matsuda, T. 1974, Gasdynamics of a centrifugal machine. *J. Fluid Mech.* **62**, 727–736.
- Spence, G. S. M., Foster, M. R. & Davies, P. A. 1992, The transient response of a contained rotating stratified fluid to impulsively started surface forcing. *J. Fluid Mech.* **243**, 33–50.
- Tassoul, J.-L. & Tassoul, M. 1990, A time dependent model for synchronization in close binaries. *Astrophys. J.* **359**, 155–163.
- Tassoul, M. & Tassoul, J.-L. 1992, On the efficiency of Ekman pumping for synchronization in close binaries. *Astrophys. J.* **395**, 604–611.
- Walín, G. 1969, Some aspects of a time-dependent motion of a stratified rotating fluid. *J. Fluid Mech.* **36**, 289–307.

## Figure Captions

**Figure 1.** The spin-up characteristic time,  $\omega$ , as a function of  $k_m N$  for varying values of  $\kappa_Y$ .

**Figure 2.** The final azimuthal velocity as a function of depth for an arbitrary value of the radius. A value of  $Z = 1$  is complete spin-up, while  $Z = 0$  is no spin-up, with  $\kappa_Y = 0$ .

**Figure 3.** As in figure 2, but with  $\kappa_Y = 10$ .

**Figure 4.** The final velocity of the central layer of the fluid ( $z = 0$ ), as a function of  $k_m N$ , for different values of  $\kappa_Y$ .

**Figure 5.** The average final spin-up of the fluid as a function of the stratification. The top two curves (solid and dashed lines) were calculated for an incompressible fluid,  $\kappa_Y = 0$ , while for the bottom two curves (dotted and dash-dotted lines)  $\kappa_Y = 10$ . With a highly compressible fluid ( $\kappa_Y = 10$ ) even very small values of  $N$  result in very little spin-up from Ekman pumping.

parameter	formula	value
$E$	$\nu_*/(2\Omega_*L_*^2)$	$10^{-7}$
$N^2$	$N_*^2/(4\Omega_*^2)$	6
$\kappa_Y$	$g_*L_*/c_{Y*}^2$	$10^2$
$F$	$4\Omega_*^2r_{c*}/g_*$	$10^{-4}$
$\Delta\kappa$	$N_*^2L_*/g_*$	$10^{-4}$
$\Omega^2$	$2\Omega_*^2L_*^2/c_{Y*}^2$	$10^{-2}$

**Table 1.** The dimensionless parameters. The values quoted are order of magnitude estimates for a characteristic pulsar.









